



# The $g$ factor of the bound muon in medium- $Z$ muonic atoms

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## ABSTRACT

We consider a theory of the  $g$  factor of a bound muon in a three-body atomic system, which consists of a spinless nucleus, a muon, and an electron in the medium range of  $Z = 10$ –30. We show that the calculation at the one-ppm level of accuracy can be separated into a consideration of an internal subsystem  $\mu - N$  (with a muon at the ground state) and an external subsystem with an electron and a compound  $\mu - N$  nucleus.

We discuss the most important contributions to the  $g$  factor of the bound muon in the  $\mu - N$  system in the medium- $Z$  approximation. In this range the list of relevant contributions contains kinematic pure Coulomb contributions, the finite-nuclear-size ones, and muonic-atom-specific contributions with closed electron loops. In the case of the medium  $Z$  one can apply a double limit  $Z\alpha \ll 1$  and  $Z\alpha m_\mu/m_e \gg 1$ , at which a number of specific contributions is simplified. Special attention is paid to non-potential contributions, i.e., those which cannot be expressed in terms of an effective potential.

We also consider the electron shielding of the magnetic moment of the compound nucleus, focusing on the corrections which are enhanced or suppressed comparing to the shielding of ordinary nuclei. In conclusion we discuss a generalization of our result for muonic atoms with a few electrons.

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## 1. Introduction

Properties of bound muons have been studied for a number of cases. In particular, the  $g$  factor of a bound muon has been measured in medium- and high- $Z$  atoms, which contain a nucleus, a muon, and electrons [1,2]. Here we consider various contributions to the  $g$  factor of a bound muon in two-body ( $\mu - N$ ) and three-body ( $e - \mu - N$ ) systems. We calculate various contributions and express some of them in terms of the related contributions to the energy levels, which simplifies their evaluation.

Muon is a heavier edition of an electron and there is a certain similarity in characteristics of muonic and electronic atoms. A large portion of the theoretical contributions to the energy levels and matrix elements has the same form for both types of the atoms. However, there are specific contributions. Those are closed-fermion-loop ones. In principle, the diagrams for them are similar for both theories, however, the character of the contributions for a loop with particles lighter than the orbiting one and those, where

the orbiting particle and the particle in the loop are the same, is very different.

Such muonic-atom-specific contributions are the electron vacuum polarization (eVP) and electron light-by-light scattering block (eLbL) ones. The dynamic parameter which determines the magnitude of the contributions is  $Z\alpha m/m_e$ . Here  $m$  is the mass of the orbiting particle (a muon in most of our considerations) and the parameter is a ratio of the characteristic atomic magnetic moment and the electron mass. (We apply here the units in which  $\hbar = c = 1$  and therefore energy, momentum and mass are expressed in the same units.)

There is an additional difference of two theories. The finite-nuclear-size (FNS) and recoil contributions are of the same form for ordinary and muonic atoms, however, they have different importance. They are enhanced in the case of muonic atoms. A number of contributions of this type can be neglected in the case of ordinary atoms, but is relevant for the muonic ones.

In the case of ordinary atoms the state of the art in calculations of the  $g$  factor of a bound electron can be found, e.g., in [3]. The contributions presented there are universal for the  $g$  factor of an electron in an ordinary hydrogenlike atom and of a muon in a two-body muonic atom. Here we focus our attention on muonic-

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**Table 1**

Parameters for some medium- $Z$  muonic atoms. The values of the rms charge radius ( $R_N$ ) are taken from [10].  $M$  stands for the nuclear mass.

| Nucleus         | $^{20}\text{Ne}$ | $^{40}\text{Ca}$ | $^{66}\text{Zn}$ |
|-----------------|------------------|------------------|------------------|
| $Z$             | 10               | 20               | 30               |
| $Z\alpha$       | 0.07             | 0.15             | 0.22             |
| $(Z\alpha)^2$   | 0.005            | 0.021            | 0.05             |
| $Z\alpha m/m_e$ | 15               | 30               | 45               |
| $R_N$ [fm]      | 3.0055(21)       | 3.4776(19)       | 3.9491(14)       |
| $mR_N$          | 1.61             | 1.86             | 2.11             |
| $Z\alpha mR_N$  | 0.12             | 0.27             | 0.46             |
| $m/M$           | 0.006            | 0.003            | 0.002            |

atom-specific contributions. We consider both the contributions with closed electronic loops and the enhanced FNS and recoil contributions.

Theory of muonic atoms is often presented in terms of numerical results. Sometimes analytic ones are possible, but they may be cumbersome [4–9]. However, there is a special limit where most of the theoretical expressions take a simple form and have a clear physical meaning. That is a medium- $Z$  limit. In this limit we can enjoy advantages of a nonrelativistic consideration, since  $Z\alpha \ll 1$ , but the eVP and eLbL contributions can be drastically simplified at the limit  $Z\alpha m/m_e \gg 1$ .

For instance, the dominant contribution to the Lamb shift in a muonic atom, which is due to the Uehling (eVP) potential, takes the form [6–8]

$$\Delta E_L(nl) = -\frac{2}{3} \frac{\alpha (Z\alpha)^2 m_r}{\pi n^2} \left[ \ln \frac{2Z\alpha m_r}{m_e n} - \sum_{k=1}^{n+l} \frac{1}{k} - \frac{5}{6} \right]. \quad (1)$$

This is a nonrelativistic contribution and all the recoil effects in this order in  $Z\alpha$  are taken into account by the use of the reduced mass  $m_r$ .

The most important muonic-atom parameters are summarized in Table 1. The value of  $Z\alpha$  does not look very small, however, for many leading nonrelativistic terms the next-to-leading correction has an extra factor of  $(Z\alpha)^2$  rather than of  $Z\alpha$ . In such atoms the value of  $mR_N > 1$  plays a role of an enhancement factor which appears in the calculation of the higher-order FNS terms. Such an enhancement makes the results of the expansion in  $Z\alpha mR_N \ll 1$  more important than the one in  $Z\alpha$ . In other words, the details of the nuclear charge distribution are more important than the relativistic corrections. We also note that in the case when the leading relativistic correction to a certain contribution has an extra  $(Z\alpha)^2$ , it is compatible with the recoil correction to that contribution.

Here we are interested in a theory of the  $g$  factor of a muon and a nucleus, bound in two-body muonic atoms. Some of our calculations are valid for an  $ns$  state, but we are mostly interested in the muon in the ground state, which reflects the experimental situation. At the first stage we consider a system of two particles bound by a potential. Such a consideration is possible in two cases. One of them is with the external-field approximation and a fully relativistic electron (described by a Dirac equation). The other possibility is with the leading nonrelativistic approximation for both particles, which eigenstates can be described by a one-body Schrödinger equation with the reduced mass  $m_r$ . Both options have been explored for pure Coulomb systems. The relativistic result was obtained in [11]. The second option was studied in detail in [12–14]. The result reads

$$g_{\mu}^{\text{Coulomb}}(ns) = g_{\mu}^{(0)} \left\{ 1 - \frac{1}{3} \left[ \left( 1 - \frac{3}{2} \frac{m}{M} \right) - a_{\mu} \left( \frac{3}{2} - \frac{5}{2} \frac{m}{M} \right) \right] \frac{(Z\alpha)^2}{n^2} \right\}, \quad (2)$$

$$g_N^{\text{Coulomb}}(ns) = g_N^{(0)} \left\{ 1 - \frac{1}{3} \left[ \left( 1 - \frac{2m}{M} \right) + \frac{1}{2(1+\kappa_N)} \frac{m}{M} \right] \frac{\alpha(Z\alpha)}{n^2} \right\}, \quad (3)$$

where  $Z$  is the nuclear charge and  $M$  is the nuclear mass,  $g_{\mu}^{(0)} = 2(1+a_{\mu})$  and  $g_N^{(0)} = 2(1+\kappa_N)$ ;  $a_{\mu}, \kappa_N$  are the anomalous magnetic moments of the muon and nucleus. Here we neglect the  $(m/M)^2$  terms, because the original evaluation in [12–14] has been performed for the nuclear spin 1/2. We are to generalize some intermediate results of [13,14] on the atomic systems bound by an arbitrary potential. While higher-order in  $m/M$  corrections in the pure Coulomb systems have been studied later in [15] (see also [16]), we rely on certain intermediate results in [13,14] and deliberately neglect here all the  $(m/M)^2$  terms.

Indeed, the atomic systems are basically bound by the Coulomb potential, however, there are numerous corrections to it. Some of them include the eVP effects, the other deal with the nuclear charge distribution etc.

Before starting a more general consideration, let us note that some eVP corrections to the pure Coulomb case can be found directly from (2) and (3) once we use the effective charge substitution

$$Z\alpha \rightarrow Z\alpha \left[ 1 + \frac{2\alpha}{3\pi} \ln \left( \frac{Z\alpha m_r}{nm_e} \right) \right], \quad (4)$$

which is valid in the case of medium- $Z$  muonic atoms.

## 2. Calculation of potential-type contributions to the $g$ factors in two-body muonic atoms

The muonic-atom-specific contributions split in two parts. One has the eVP loops and can be expressed in terms of a certain binding potential, the others deal with the eLbL loop and cannot be expressed through such a modification. The former are considered in this section, while the latter are studied in the next one. We refer to the former as ‘potential-type’ contributions. Such contributions have been studied in [17] and [18] for the muon  $g$  factor in the nonrecoil limit, while the recoil corrections and contributions to the nuclear  $g$  factor have been considered by us in [19].

Let us consider a two-body muonic atom bound by a central binding potential  $V(r)$ . Calculation of the contributions to the  $g$  factor of two particles bound by an arbitrary central potential can be performed starting with the identities [13] (cf. [14])

$$g_{\mu}^{\text{bound}}(ns) = g_{\mu}^{(0)} \left\{ 1 - \frac{1}{3} \left[ 1 - \frac{3a_{\mu}}{2(1+a_{\mu})} + \frac{1}{2(1+a_{\mu})} \frac{m}{M} \right] \frac{\langle \mathbf{p}^2 \rangle}{m^2} \right\}, \quad (5)$$

$$g_N^{\text{bound}}(ns) = g_N^{(0)} \left\{ 1 - \frac{1}{3Z} \left[ 1 + \frac{1}{2(1+\kappa_N)} \frac{m}{M} \right] \frac{\langle \mathbf{p}^2 \rangle}{m^2} \right\}. \quad (6)$$

Those identities can be further transformed to [19] (cf. [18])

$$g_{\mu}^{\text{bound}}(ns) = g_{\mu}^{(0)} \left\{ 1 + \frac{2}{3} \left[ 1 - \frac{3a_{\mu}}{2(1+a_{\mu})} + \frac{1}{2(1+a_{\mu})} \frac{m}{M} \right] \times \left( 1 - \frac{2m}{M} \right) \frac{\partial E}{\partial m_r} \right\}, \quad (7)$$

$$g_N^{\text{bound}}(ns) = g_N^{(0)} \left\{ 1 + \frac{2}{3Z} \left[ 1 + \frac{1}{2(1+\kappa_N)} \frac{m}{M} \right] \times \left( 1 - \frac{2m}{M} \right) \frac{\partial E}{\partial m_r} \right\}. \quad (8)$$

Here,  $E$  is the nonrelativistic energy in the center-of-mass system of the two-body atom and as such it can be expressed in terms

of the reduced mass  $m_r$ . In nonrelativistic physics the use of the reduced mass allows us to take into account the recoil effects.

Until recently, only the terms, related to  $g_\mu^{\text{bound}}$  at the external-field limit, have been known for an arbitrary potential [17,18].

The application of those identities is very straightforward. Let us consider some examples of the application of (7) and (8) to the medium- $Z$  case. For instance, we know at medium  $Z$  the result for the eVP correction to the energy (see (1)), which immediately leads to the result (cf. [20,19])

$$g_\mu^{\text{eVP}}(ns) = -\frac{4}{9} \frac{\alpha(Z\alpha)^2}{n^2} g_\mu^{(0)} \left(1 - \frac{3}{2} \frac{m}{M}\right) \times \left[ \ln \frac{2Z\alpha m_r}{m_e n} - \sum_1^n \frac{1}{k} + \frac{1}{6} \right], \quad (9)$$

$$g_N^{\text{eVP}}(ns) = -\frac{4}{9} \frac{\alpha^2(Z\alpha)}{n^2} g_N^{(0)} \left[1 + \left(\frac{1}{2(1+\kappa_N)} - 2\right) \frac{m}{M}\right] \times \left[ \ln \frac{2Z\alpha m_r}{m_e n} - \sum_1^n \frac{1}{k} + \frac{1}{6} \right]. \quad (10)$$

Let us now consider the FNS contributions. The leading one follows from the well-known identity for the FNS contribution to the energy levels

$$\Delta E_{\text{fns:lead}}(ns) = \frac{2}{3n^3} (Z\alpha)^4 m_r^3 R_N^2. \quad (11)$$

As mentioned above, the value of  $mR_N$  serves as an enhancement parameter and therefore we should expect that a  $Z\alpha mR_N$  correction to the leading FNS term (9) is larger than a  $Z\alpha$  one. Actually, the linear  $Z\alpha$  correction is absent and the relativistic correction to the leading term has an extra factor of  $(Z\alpha)^2$ . To find the  $Z\alpha mR_N$  correction to the leading FNS contribution to the bound  $g$  factors it is sufficient to consider the related  $Z\alpha mR_N$  correction to the energy. That is a so-called Friar term which reads [21,22]

$$\Delta E_{\text{fns:3}}(ns) = -\frac{(Z\alpha)^5 m_r^4}{3n^3} \langle r^3 \rangle_2, \quad (12)$$

where

$$\langle r^3 \rangle_2 \equiv \int d^3r d^3r' \rho_E(\mathbf{r}) \rho_E(\mathbf{r}') |\mathbf{r} - \mathbf{r}'|^3 \quad (13)$$

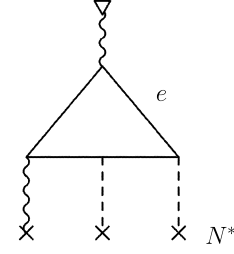
and  $\rho_E(\mathbf{r})$  is the nuclear-charge distribution.

Combining (11) and (12), we obtain from (7) and (8) (cf. [23,18,19])

$$g_\mu^{\text{fns}}(ns) = \frac{4(Z\alpha)^4 (m_r R_N)^2}{3n^3} g_\mu^{(0)} \left[1 - \frac{3a_\mu}{2} - \frac{3}{2} \frac{m}{M}\right] \times \left[1 - \frac{2Z\alpha m_r \langle r^3 \rangle_2}{3R_N^2}\right], \quad (14)$$

$$g_N^{\text{fns}}(ns) = \frac{4\alpha(Z\alpha)^3 (m_r R_N)^2}{3n^3} g_N^{(0)} \left[1 + \left(\frac{1}{2(1+\kappa_N)} - 2\right) \frac{m}{M}\right] \times \left[1 - \frac{2Z\alpha m_r \langle r^3 \rangle_2}{3R_N^2}\right]. \quad (15)$$

We discuss the numerical values of some contributions in Sect. 4.



**Fig. 1.** The contribution in order  $\alpha(Z\alpha)^2$  to the bound  $g$  factor of a nucleus with the nuclear charge  $Z$ . In the original paper [26] the nucleus was considered as a 'true' one. Here, we are interested in a consideration of an 'effective' nucleus  $N^*$ , which is a closely bound  $\mu - N$  subsystem.

### 3. Specific non-potential contributions to the $g$ factors in two-body muonic atoms

The muonic-atom-specific contributions involve closed electron loops. The eVP loops induce a correction to the Coulomb binding potential and the related effects have been considered above. In this section we consider the effects of the electron-loop light-by-light scattering. Similar diagrams have been considered for ordinary atoms in [24,25], however, the case of eLbL in muonic atoms has its specifics. Actually, there has been consideration which can be adjusted to medium- $Z$  muonic atoms. Some time ago Milstein and Yelkhovsky [26] considered a contribution to the anomalous magnetic moment of a nucleus (as a compound particle) due to eLbL (see Fig. 1).

The result of [26] (cf. [27]), which reads

$$\frac{\Delta g}{g^{(0)}} = \frac{2}{3} \frac{\alpha(Z\alpha)^2}{\pi} \left(1 + 0.657(Z\alpha)^2\right) \ln \frac{1}{m_e R_N}, \quad (16)$$

is present in terms of a regularized logarithmic divergence and a coefficient in front of it. The divergence there is cut by the nuclear size, which for ordinary nuclei with a logarithmic accuracy can be substituted by the rms charge radius  $R_N$ . For the sake of simplicity we keep for further numerical estimations only the leading in  $Z\alpha$  term in (16).

To adjust the result of (16) to the  $\mu - N$  subsystem as a compound nucleus  $N^*$ , one has to take into account a few differences between the 'standard' nucleus  $N$  and the effective one  $N^*$ .

(i) The nuclear charge of the  $\mu - N$  subsystem is equal to  $Z - 1$ .

(ii) In the case of the  $\mu - N$  system the 'geometrical' size and  $R_N$  differ significantly because the distributed charge is a small portion of the total one. The size is  $R_{\text{geom}} = \sqrt{3}/(Z\alpha m_r)$ , while the rms radius is  $R_N^2 \approx -R_{\text{geom}}^2/(Z - 1)$  (cf. [28,29]). Since we need a logarithmic accuracy,  $(Z\alpha m_r)^{-1}$  is used as a logarithmic cut-off.

(iii) The compound-nuclear magnetic moment is composed of the muon magnetic moment and the nuclear one. The expression is linear in the nuclear magnetic moment and we can consider separately corrections to the muon and nuclear  $g$  factors.

(iv) The consideration in [26] deals with the situation when there is only one source of the logarithmic divergence in the external field approximation with point-like nuclei. The related result would be with a coefficient  $2/3(Z - 1)^2\alpha^3/\pi$ . In our case the situation is somewhat different. In particular, there is a logarithmic divergence in the anomalous magnetic moment of muon [27] with a coefficient  $2/3\alpha^3/\pi$ . This cut off is not due to the nuclear size, but to the muon recoil and is already included into our calculation (since we consider in our equations a complete value of  $g_\mu^{(0)}$ ). Therefore we have to subtract  $2/3\alpha^3/\pi$  from  $2/3(Z - 1)^2\alpha^3/\pi$ . The remaining logarithmic contribution reads (cf. [30])

**Table 2**

Contributions to the bound  $g_\mu$  factor in the ppm units for  $g_\mu/g_\mu^{(0)}$ : the  $\mu - N$  subsystem with the muon in the ground state.

| Contribution, Ref.                    | $^{20}\text{Ne}$ | $^{40}\text{Ca}$    | $^{66}\text{Zn}$    |
|---------------------------------------|------------------|---------------------|---------------------|
| Coulomb, $(Z\alpha)^2$ , (2)          | −1775.0          | −7100.2             | −15 975.4           |
| Coulomb, $(Z\alpha)^2(m/M)$ , (2)     | 15.1             | 30.2                | 41.2                |
| Coulomb, $a_\mu(Z\alpha)^2$ , (2)     | 3.1              | 12.4                | 27.9                |
| Coulomb, $(m/M)^2$ , (20)             | −0.9             | −1.8                | −2.2                |
| Coulomb, $(Z\alpha)^4$ , (19)         | −2.4             | −37.8               | −191.4              |
| Coulomb, $(Z\alpha)^{4+}(m/M)$ , [35] | 0.04             | 0.5                 | 1.7                 |
| FNS:lead, (14)                        | 98               | $2.1 \times 10^3$   | $13.7 \times 10^3$  |
| FNS:lead-recoil, (14)                 | −1.9             | −21                 | −82                 |
| $a_\mu \times$ FNS:lead, (14)         | −0.2             | −3.7                | −24                 |
| FNS: $(mR_N)^3$ , (14)                | −25.1            | $−1.25 \times 10^3$ | $−13.9 \times 10^3$ |
| eVP at $m/M = 0$ , (9)                | −44.5            | −226                | −571                |
| eVP-recoil, (9)                       | 0.5              | 1.2                 | 1.7                 |
| eLbL, (17)                            | 18               | $1.0 \times 10^2$   | $2.6 \times 10^2$   |
| One-loop SE, (21)                     | 0.3              | 1.8                 | 1.5                 |

$$\frac{\Delta g_\mu(1s)}{g_\mu^{(0)}} = \frac{2}{3} \frac{Z(Z-2)\alpha^3}{\pi} \ln \frac{Z\alpha m_\mu}{m_e}. \quad (17)$$

(v) The logarithmic correction to the *free* magnetic moment of the nucleus  $N$  with charge  $Z$  follows (16) and should also be subtracted. The result for the bound nuclear  $g$  factor is therefore

$$\frac{\Delta g_N(1s)}{g_N^{(0)}} = -\frac{2}{3} \frac{(2Z-1)\alpha^3}{\pi} \ln \frac{Z\alpha m_\mu}{m_e}. \quad (18)$$

We discuss the numerical values of some contributions in Sect. 4.

#### 4. Three-body muonic atoms

Let us consider now a three-body system  $e - \mu - N$  with a spinless nucleus  $N$ , which is the simplest one to account for the most important corrections to the value of the  $g$  factor of a bound muon. The  $g$  factor of the muon, bound by the nucleus, has the standard corrections as in a two-body  $\mu - N$  system. In the meantime, the magnetic moment of such a compact two-body system is shielded by an electron, bound by that ‘compound nucleus’. We consider the contributions as factorized ones and we discuss accuracy of such an hierarchy approach after the consideration of the numerical values of the contributions is done. In principle, such an approach is a ‘good’ one and has been successfully applied to the hyperfine  $e - \mu$  interval [31–34] and to the Lamb shift of the electronic states [28] (both in the neutral muonic helium). The practical question is the accuracy of such an approach, which is better to discuss once we establish the magnitude of various corrections.

Hence we consider the hierarchy three-body system through a subsequent consideration of two two-body subsystems. The internal one,  $N^* = \mu - N$ , plays a role of an effective nucleus for the external subsystem  $e - N^*$ . We start our consideration with the internal subsystem. We are interested in contributions to the value of the bound muon  $g$  factor, which has been already studied in two previous sections. We summarize these results in Table 2 for three medium- $Z$  muonic atoms, parameters of which are listed in Table 1.

We present there numerical results for various contributions. Some of them have been considered above in detail, while others have been known for a while, but have not been discussed above. Expressions for some of the contributions, such as the eVP ones, have been drastically simplified in the medium- $Z$  limit. We have

also found above a simple presentation for the eLbL contributions. The few first contributions follow from a consideration of a pure Coulomb system. Three of them are given in (2) following [12–14]. The higher-order external-field relativistic correction was found in [11]

$$g_\mu^{\text{Dirac}}(1s) = 2 \left[ 1 - \frac{1}{3}(Z\alpha)^2 - \frac{1}{12}(Z\alpha)^4 + \dots \right]. \quad (19)$$

The approach in [12–14] allows to find the  $(Z\alpha)^2(m/M)$  recoil contribution. The higher order in the  $m/M$  recoil correction

$$g_\mu(1s) = -g_\mu^{(0)} \times \frac{(Z+1)(Z\alpha)^2}{2} \left( \frac{m_\mu}{M} \right)^2 \quad (20)$$

was obtained in [15] (cf. [16]), while a higher-order in the  $Z\alpha$  one was obtained in [35]. Note that the  $(Z\alpha)^4(m/M)$  coefficient is known analytically [36,37,35], however, it is small and the  $(Z\alpha)^5(m/M)$  [35] contribution is larger than the  $(Z\alpha)^4(m/M)$  one [36,37,35] for the atoms of interest. For this reason we refer to that correction as the  $(Z\alpha)^{4+}(m/M)$  one.

The other group of contributions is due to the nuclear-structure effects. Accuracy of two largest contributions is very limited and the result strongly depends on the details of the nuclear shape. The  $(mR_N)^3$  term is due to the Friar contribution to the energy in (12), which for our estimation is calculated with the homogeneous-sphere nuclear-charge distribution, but may be evaluated with a more realistic model. We note that this contribution is compatible with the leading FNS term. That is because the parameter  $Z\alpha mR_N$  is somewhat below but comparable with unity (see Table 1), which means that the atomic size is comparable with the nuclear size. In this case details of the charge distribution are very important. The calculation of the  $(mR_N)^3$  term obviously depends on the model of the distribution. The leading FNS term is expressed directly through  $R_N$  as in (14) in a model-independent way. However, to find an accurate value of the rms nuclear charge is possible only through an application of a certain model. While at the low end of medium  $Z$  the FNS terms are under control at the level of a few ppm, at the high end of that range ( $Z = 30$ ) the systematic uncertainty due to the nuclear shape may exceed 100 ppm. The nuclear-polarizability contribution is not included into our consideration. It strongly depends on the nucleus and does not allow us any general numerical estimations and in principle may exceed the level of  $10^3$  ppm.

The third group of the contributions is due to the eVP and eLbL ones. They are found for a pointlike nucleus (as found in (9)), which seems quite reasonable for  $Z = 10$ , but may be questioned for  $Z = 30$ .

The last line is due to a universal correction, which is not muonic-atom specific. That is a [muon] self-energy contribution [38]. The ‘kinematic’ part of the one-loop corrections of order  $\alpha(Z\alpha)^2$  which comes from the anomalous magnetic moment contribution to (2) is excluded from the one-loop self-energy contribution. The result with the  $a_\mu$  part subtracted is [38]

$$g_\mu(1s) = \frac{\alpha}{\pi} (Z\alpha)^4 g_\mu^{(0)} \left[ \frac{16}{9} \ln \frac{1}{(Z\alpha)^2} - 5.118\dots \right]. \quad (21)$$

That concludes our consideration of the  $\mu - N$  subsystem and now we turn to the external  $e - N^*$  subsystem, considering shielding of the muon magnetic moment by the electron.

Some of the contributions of interest have been considered above, but we have to take into account that the effective compound nucleus somewhat differs from a standard one. For instance, we have already discussed above the size and the value of the rms



**Table 3**

Contributions to the electron shielding of the muon magnetic moment in the ppm units for  $g_\mu/g_\mu^{(0)}$  for the electron and muon in their ground state. We remind that while using the standard equation (3) one has to recall that the nuclear charge is  $Z - 1$  etc.

| Contribution  | $^{20}\text{Ne}$      | $^{40}\text{Ca}$      | $^{66}\text{Zn}$      |
|---|-----------------------|-----------------------|-----------------------|
| Coulomb, $\alpha(Z\alpha)$ , (3)                                    | −159.7                | −337                  | −515                  |
| Coulomb, $\alpha(Z\alpha)(m/M)$<br>at $\kappa_{N^*} = \infty$ , (3) | $-0.9 \times 10^{-2}$ | $-0.9 \times 10^{-2}$ | $-0.9 \times 10^{-2}$ |
| Coulomb- $\kappa_{N^*}$ , (3) & (23)                                | $1 \times 10^{-3}$    | $1.2 \times 10^{-3}$  | $1 \times 10^{-3}$    |
| Coulomb, $\alpha(Z\alpha)^3$ , (24)                                 | −1.9                  | −17.5                 | −62.1                 |
| FNS:lead, (22)  | −0.004                | −0.004                | −0.005                |

charge radius  $R_N$ . Following [28,29] and (15), we find the leading FNS contribution for the  $e - N^*$  system

$$\Delta g_\mu(ns) = -\frac{4(Z-1)^2\alpha^2}{Z^2n^3} \left(\frac{m_e}{m_\mu}\right)^2 g_\mu^{(0)}. \quad (22)$$

The compound nucleus  $N^*$  also possesses a large value of the anomalous magnetic moment, which enters (3). That is because the own magneton is  $\sim (Z-1)e/M \sim Ze/Am_p$ , while the magnetic moment is basically the muon one  $\sim -e/2m_\mu$  (here  $A$  is the relative atomic weight). The factor in (3) related to the anomalous magnetic moment is [19]

$$(1 + \kappa_{N^*}) \frac{(Z-1)e}{M + m_\mu} = -\frac{e}{m_\mu},$$

where  $e$  stands for the [positive] elementary charge. The required factor is

$$\frac{1}{1 + \kappa_{N^*}} \simeq -\frac{(Z-1)m_\mu}{M + m_\mu}. \quad (23)$$

Related numerical results for the shielding factor are summarized in Table 3. Four first are for a pure Coulomb shielding factor in (3). Shielding at  $\kappa_{N^*} = \infty$  is related to the atoms when the *anomalous magnetic moment* of the nucleus dominates over the Dirac's value. That is the case when the relatively large value of the muon magnetic moment determines the magnetic moment of a particle which has a charge and mass as a standard nucleus. The correction with  $\kappa_{N^*}$  is suppressed in contrast to that in an ordinary atom, however, it is a recoil correction of order  $\alpha(Z\alpha)(m/M)$  and its influence is marginal. The theory of shielding even in the case of a few electrons seems to be an external-field theory at the level of one ppm.

Among the terms discussed above we also listed there a higher-order external-field shielding correction [39–41], which for the 1s state takes the form

$$g_N^{\text{bound}}(1s) = g_N^{(0)} \left[ 1 - \frac{\alpha(Z\alpha)}{3} - \frac{97}{108} \alpha(Z\alpha)^3 \right]. \quad (24)$$

The results in Table 3 are given for the electron 1s state. As to an  $ns$  state of the electron, the leading Coulomb contributions (including the recoil) to the  $g$  factor of the bound nucleus scale as  $n^{-2}$  [12–14]. The FNS term scales as  $n^{-3}$  [19]. The Coulomb shielding  $\alpha(Z\alpha)^3$  contribution has an overall  $n^{-3}$  factor, but also a somewhat more complicated  $n$  dependence [39–41].

The equation (15) allows us to take into account the  $(m_e R_{N^*})^3$  term. Comparing with the leading FNS correction it has an extra  $(m_e/m_\mu)$  factor (which comes as a ratio of atomic and compound-nuclear sizes) and  $1/(Z-1)$  (since the distributed charge is the muon one). Such a correction is obviously much smaller than the FNS leading term. The compound-nuclear-polarizability contribution is smaller than the leading FNS terms and larger than the  $(m_e R_{N^*})^3$  term (cf. [42]).

The specific values of the nuclear charge and anomalous magnetic moment present the leading three-body contributions, i.e., the contributions which simultaneously involve parameters of the internal and external subsystems. They are essentially below one-ppm level. The compound-nuclear-polarizability contribution is a dynamic three-body contribution, which is subdominant, being smaller than two mentioned above contributions. Its smallness completely validates the hierarchy approach with a subsequent consideration of the internal and external two-body subsystems.

## 5. Conclusions

The purpose of the paper is to sort out which contributions to the  $g$  factor of a bound muon in a few-body muonic atomic systems (nucleus-muon-electrons) are the most important and to find an efficient way to estimate them. A number of the contributions have been discussed above for the simplistic case of a single electron and a spinless nucleus.

Some of them may be improved in a relatively easy way. However, before doing that one has to clearly understand the importance of various terms. The experimental data are available on a very few elements [1,2]. The accuracy is at the level of 10 ppm. We hope that the variety of the elements can be extended and possibly the accuracy can be somewhat improved. To improve the theory for a particular element we have to specify it first and next improve the accuracy.

One of the crucial problems is the improvement of the accuracy of the electron shielding in an actual experimental situation, where atoms have many electrons. An accurate theory of the shielding factor for the nuclear magnetic moment in an ordinary atom is required if one intends to use a value of this magnetic moment for other calculations, such as for the hyperfine interval in a hydrogen-like ion. The verification of a theory for the shielding factor is a hard and complicated issue. A successful theory of the bound-muon  $g$  factor opens an opportunity to use the muonic atoms to verify the theory of electron shielding. In the case of the compound nucleus  $N^* = \mu - N$ , one meets an opportunity to measure the shielded magnetic moment of the 'nucleus', for which the free magnetic moment is known.

The isotopic effects in shielding for the true nucleus with the charge  $Z - 1$  and the compound nucleus, built of a nucleus with the charge  $Z$  and a muon, seem negligible at the level of 10 ppm. The ratio of magnetic moments of the bound muon and the true  $Z - 1$  nucleus is to a large extent free of multielectron atomic-physics calculations. The situation with isotopic effects has a number of other options. One may consider several isotopes of the same element (both for muonic  $Z$  atoms and their pure electronic  $Z - 1$  counterpart). It may be also possible to consider the muonic atoms with a nucleus with a spin and to compare the magnetic moment of a bound muon and a bound nucleus.

An important conclusion for the isotopic effects, which follows from the consideration of three-body systems, is that likely in muonic atoms with the large number of electrons, shielding of the muon magnetic moment is essentially the same as of a true  $Z - 1$  nucleus. A part of the recoil contribution due to the [compound] nuclear anomalous magnetic moment is suppressed (comparing with an ordinary  $Z - 1$  nucleus), while the finite-nuclear-size and the nuclear-polarizability ones are enhanced, but it seems that all of them are below a one-ppm level even in the case of many electrons. Eventually at the one-ppm level one has to consider the shielding within the external field approximation and for a point-like nucleus.

Such a situation validates a consideration of a two-body internal subsystem  $\mu - N$  independently of the electrons. A good side of studies of the  $\mu - N$  systems is that at the one-ppm level one

can ignore the effects, which are the most important for the theory of the  $g$  factor or a bound electron in a hydrogenlike atom, which are due to the one-loop and two-loop QED radiative corrections. The theory is completely based on the muonic-atom-specific contributions and on an accurate account of the recoil effects.

At the low end of the medium- $Z$  interval ( $Z = 10$ ) the situation with the theory at the ten-ppm level is rather an acceptable one and at the one-ppm level it seems promising. On the contrary, at the high end of the medium- $Z$  interval ( $Z = 30$ ) the situation is far from being perfect. The nuclear polarizability in the muonic atom is to be studied. Instead of a pure Coulomb contribution one has to deal with a nuclear-shape model and to perform a numerical solution of the bound equations. With the atomic size (for the electronic states) is comparable with the size of the compound nucleus, the numerical model-dependent approaches seem unavoidable.

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